On Beltrami Model of de Sitter Spacetime

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Based on some important properties of dS space, we present a Beltrami model $\mathcal{B}_\Lambda$ that may shed light on the observable puzzle of dS space and the paradox between the special relativity principle and cosmological principle. In $\mathcal{B}_\Lambda$, there are inertial-type coordinates and inertial-type observers. Thus, the classical observables can be defined for test particles and light signals. In addition, by choosing the definition of simultaneity the Beltrami metric is transformed to the Robertson-Walker-like metric. It is of positive spatial curvature of order Λ. This is more or less indicated already by the CMB power spectrum from WMAP and should be further confirmed by its data in large scale.

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I. INTRODUCTION

Among many puzzles in ordinary approach to dS space [1], one is how to define observables on it [2]. There has been also a long-standing paradox between the special relativity (SR) principle and the cosmological principle [3], called the SRP-CP paradox, since cosmological models were proposed and especially since the cosmic microwave background (CMB) was discovered. Roughly speaking, it may be simply described as: "what would happen for SR, if the CMB could be discovered before 1905?"

In this letter, we present a proposal to the dS observable puzzle and the SRP-CP paradox. The key observation is based upon some simple but important properties of dS space [4]–[9]. In fact, among various metrics of dS spaces, there is an important one in which dS space is in analog with Minkowski space. It is the dS space with Beltrami-like metric, called BdS space and denoted as $\mathcal{B}_\Lambda$. BdS space is precisely the Beltrami-like model [10] of a 4-hyperboloid $\mathcal{S}_\Lambda$ in 5-d Minkowski space, i.e. $\mathcal{B}_\Lambda \simeq \mathcal{S}_\Lambda$. In $\mathcal{B}_\Lambda$ there exist a set of Beltrami coordinate systems, which covers $\mathcal{B}_\Lambda$ patch by patch, and in which test particles and light signals move along the timelike and null geodesics, respectively, with *constant* coordinate velocity. Therefore, they look like in free motion in a spacetime without gravity. Thus, the Beltrami coordinates and observers $\mathcal{O}_B$ at these systems may be regarded as of "global inertial-type". And the classical observables for these particles and signals may be well defined. Thus, it may shed light on the dS observable puzzle.

Why in a constant curvature spacetime do there exist such motions and observers of global inertial-type? As is well known, if we start with the 4-d Euclidean geometry and weaken the fifth axiom, then there should be 4-d Riemann, Euclid, and Lobachevski geometries at equal footing. Importantly, their geodesics are globally straight lines in certain coordinate systems that are just ones in analog with the coordinate systems in Beltrami model of Lobachevski plane [10], and under corresponding transformation groups the systems transform among themselves. Now changing the signature to -2, these constant curvature spaces turn to dS, Minkowski, and AdS spacetimes, respectively, and those straight lines are classified by timelike, null and spacelike straight world-lines. Thus, in analog with SR, the former two should describe the "global inertial-type motions" for free particles and light signals, respectively. And, Einstein's SR principle should also be available in $\mathcal{B}_\Lambda$, called SR-type principle with respect to dS, Poincarè, AdS group, respectively.

If the Beltrami coordinates make sense, they should concern with the measurements in laboratory at one patch and simultaneity should be defined with respect to the time coordinate. On the other hand, the simultaneity may also be defined by the proper time of a clock rest at the origin of Beltrami spatial coordinates. Thus, there are two kinds of simultaneity in $\mathcal{B}_\Lambda$. Consequently, if the simultaneity is transformed from the first to the second, the Beltrami metric is reduced to the Robertson-Walker(RW)-like metric in $\mathcal{B}_\Lambda$ with positive, rather than zero or negative, spatial curvature and its deviation from zero is in the order of cosmological constant Λ. Thus, the second simultaneity should link with the observation in the cosmological scale. This important property differs with either SR where two of them coincide, or usual cosmological models where k is a free parameter. Thus, this property sheds light on the SRP-CP paradox. In addition, the tiny spatial closeness seems more or less to have been already confirmed by the CMB power spectrum data from WMAP [11] and should be further checked by its data in large scale.

This letter is organized as follows. In section 2, we briefly set up a framework for $\mathcal{B}_\Lambda$. In section 3 we study the motion of particles and light signals and define their observables. In section 4, we define two kinds of simultaneity, the length of a ruler, and so on. We also show

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that $\mathcal{B}_\Lambda$ can be reduced to RW-like metric with a slightly closed space. Finally, we end with a few remarks.

II. THE BELTRAMI-dS SPACETIME

We start with a 4-d hyperboloid $\mathcal{S}_\Lambda$ embedded in a 5-d Minkowski space with $\eta_{AB} = \mathrm{diag}(1,-1,-1,-1,-1)$:

$$\mathcal{S}_\Lambda: \quad \eta_{AB}\xi^A\xi^B = -R^2, \tag{1}$$
$$ds^2 = \eta_{AB}d\xi^A d\xi^B, \tag{2}$$

where $R^2 := 3\Lambda^{-1}$ and $A, B = 0, \ldots, 4$. Clearly, Eqs. (1) and (2) are invariant under dS group $\mathcal{G}_\Lambda = SO(1,4)$.

The Beltrami coordinates are defined patch by patch on $\mathcal{B}_\Lambda \simeq \mathcal{S}_\Lambda$. For intrinsic geometry of $\mathcal{B}_\Lambda$, there are at least eight patches $U_{\pm\alpha} := \{\xi \in \mathcal{S}_\Lambda : \xi^\alpha \gtrless 0\}, \alpha = 1, \cdots, 4$. In $U_{\pm 4}$, for instance, the Beltrami coordinates are

$$x^i|_{U_{\pm 4}} = R\xi^i/\xi^4, \qquad i = 0, \cdots, 3; \tag{3}$$
$$\xi^4 = \pm({\xi^0}^2 - \sum_{a=1}^{3}{\xi^a}^2 + R^2)^{1/2} \neq 0. \tag{4}$$

In the patches $\{U_{\pm a}, a = 1, 2, 3\}$,

$$y^{j'}|_{U_{\pm a}} = R\xi^{j'}/\xi^a, \quad j' = 0, \cdots, \hat{a} \cdots, 4; \quad \xi^a \neq 0, \tag{5}$$

where $\hat{a}$ means omission of a. It is important that all transition functions in intersections are of $\mathcal{G}_\Lambda$, say, in $U_4 \bigcap U_3$, the transition function $T_{4,3} = \xi^3/\xi^4 = x^3/R = R/y^4 \in \mathcal{G}_\Lambda$ so that $x^i = T_{4,3}y^{i'}$.

In each patch, there are condition and Beltrami metric

$$\sigma(x) = \sigma(x,x) := 1 - R^{-2}\eta_{ij}x^i x^j > 0, \tag{6}$$
$$ds^2 = [\eta_{ij}\sigma(x)^{-1} + R^{-2}\eta_{ik}\eta_{jl}x^k x^l \sigma(x)^{-2}]dx^i dx^j. \tag{7}$$

Under fractional linear transformations of $\mathcal{G}_\Lambda$

$$\begin{array}{l} x^i \to \tilde{x}^i = \pm\sigma(a)^{1/2}\sigma(a,x)^{-1}(x^j - a^j)D^i_j, \\ D^i_j = L^i_j + R^{-2}\eta_{jk}a^k a^l(\sigma(a) + \sigma(a)^{1/2})^{-1}L^i_l, \\ L := (L^i_j)_{i,j=0,\cdots,3} \in SO(1,3), \end{array} \tag{8}$$

where $\eta_{ij} = \mathrm{diag}(1,-1,-1,-1)$ in $U_{\pm\alpha}$, Eqs. (6) and (7) are invariant.

Note that Eqs. (6)-(8) are defined on $\mathcal{B}_\Lambda$ patch by patch. This is, in fact, a cornerstone for the SR-type principle. $\sigma(x) = 0$ may be considered as the boundary of BdS spacetime, $\partial\mathcal{B}_\Lambda$.

For two separate events $A(a^i)$ and $X(x^i)$ in $\mathcal{B}_\Lambda$,

$$\Delta^2_\Lambda(A,X) = R[\sigma^{-1}(a)\sigma^{-1}(x)\sigma^2(a,x) - 1] \tag{9}$$

is invariant under $\mathcal{G}_\Lambda$. Thus, the interval between A and B is timelike, null, or spacelike, respectively, according to

$$\Delta^2_\Lambda(A,B) \gtreqless 0. \tag{10}$$

The proper length of timelike or spacelike between A and B are integral of $\mathcal{I}ds$ over the geodesic segment $\overline{AB}$:

$$S_{timelike}(A,B) = R\sinh^{-1}(|\Delta(a,b)|/R), \tag{11}$$
$$S_{spacelike}(A,B) = R\arcsin(|\Delta(a,b)|/R), \tag{12}$$

where $\mathcal{I} = 1, -i$ for timelike or spacelike, respectively.

It can be shown that the light-cone at A with running points X is

$$\mathcal{F}_\Lambda := R\{\sigma(a,x) \mp [\sigma(a)\sigma(x)]^{1/2}\} = 0. \tag{13}$$

It satisfies the null-hypersurface condition

$$g^{ij}\frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j}\bigg|_{f=0} = 0, \tag{14}$$

where $g^{ij} = \sigma(x)(\eta^{ij} - R^{-2}x^i x^j)$ is the inverse Beltrami metric.

III. MOTION AND OBSERVABLES OF TEST PARTICLES

We now show that in $\mathcal{B}_\Lambda$ the geodesics are Lobachevski-like straight world lines, along which the observables for test particles and signals may be well defined.

For a free particle with mass $m_{\Lambda 0}$, it should move globally along a timelike geodesic with respect to the Beltrami metric. It is easy to show that the geodesic equation is equivalent to

$$\frac{dp^i}{ds} = 0, \quad p^i := m_{\Lambda 0}\sigma(x)^{-1}\frac{dx^i}{ds} = C^i = \mathrm{const.} \tag{15}$$

This implies that under the initial condition

$$x^i(s=0) = b^i, \qquad \frac{dx^i}{ds}(s=0) = c^i$$

with the constraint

$$g_{ij}(b)c^i c^j = 1,$$

a new parameter $w = w(s)$ can be chosen such that the geodesic is just a straight world-line

$$x^i(w) = c^i w + b^i. \tag{16}$$

This property is in analog with the straight line in the Beltrami model of Lobachevski plane. The parameter w can be integrated out,

$$w(s) = \begin{cases} Re^{\mp s/R}\sinh\dfrac{s}{R}, & \eta_{ij}\,c^i c^j = 0, \\ \dfrac{R\sinh\dfrac{s}{R}}{\dfrac{\eta_{ij}\,c^i b^j}{R\sigma(b)}\sinh\dfrac{s}{R} + \cosh\dfrac{s}{R}}, & \eta_{ij}\,c^i c^j \neq 0. \end{cases} \tag{17}$$

Similarly, a light signal moves globally along a null geodesic. The null geodesic equation formally still has the first integration

$$\sigma^{-1}(x)\frac{dx^i}{d\tau} = \mathrm{constant}, \tag{18}$$

but now the condition $ds = 0$ also holds, where τ is an affine parameter. Again, under the initial condition

$$x^i(\tau = 0) = b^i, \qquad \frac{dx^i}{d\tau}(\tau = 0) = c^i. \tag{19}$$

and the constraint

$$g_{ij}(b)\, c^i c^j = 0, \tag{20}$$

the null geodesic can be expressed as a straight line

$$x^i = c^i w(\tau) + b^i,$$

where

$$w(\tau) = \begin{cases} \tau, & \eta_{ij}\, c^i c^j = 0, \\ -\dfrac{R^2\sigma(b)}{|\eta_{ij} c^i c^j|}\left(\dfrac{1}{\tau+\tau_0} - \dfrac{1}{\tau_0}\right), & \eta_{ij}\, c^i c^j \neq 0, \end{cases} \tag{21}$$

with

$$\tau_0 = \sqrt{\frac{R^2\sigma(b)}{|\eta_{ij} c^i c^j|}}.$$

Thus, for free particle and light signal the components of the coordinate velocity are constants, respectively:

$$\frac{dx^a}{dt} = v^a; \quad \frac{d^2x^a}{dt^2} = 0; \quad a = 1, 2, 3. \tag{22}$$

Of course, this makes sense only if the Beltrami coordinate system is of physical meaning as inertial-type.

Now we are ready to define the observables for test particles. From the (15), it is natural to define the conservative quantities p^i along the geodesic as the 4-momentum of a free particle with mass $m_{\Lambda,0}$ and its zeroth component as the energy. Note that this 4-momentum is no longer a 4-vector rather a pseudo 4-vector.

Furthermore, for a free particle a set of quantities L^{ij} may also be defined by the following equation and they are also conserved along a geodesic

$$L^{ij} = x^i p^j - x^j p^i; \quad \frac{dL^{ij}}{ds} = 0. \tag{23}$$

These may be called the 4-angular-momentum of a particle and they are also no longer the components of an anti-symmetric tensor but a pseudo anti-symmetric tensor. However, p^i and L^{ij} constitute a 5-d angular momentum for a free particle in $\mathcal{S}_\Lambda$ and it is conserved along the geodesics that are the straight world-line

$$\mathcal{L}^{AB} := m_{\Lambda 0}(\xi^A \frac{d\xi^B}{ds} - \xi^B \frac{d\xi^A}{ds}); \quad \frac{d\mathcal{L}^{AB}}{ds} = 0. \tag{24}$$

And Einstein's famous formula for such a kind of free particles can be generalized in $\mathcal{B}_\Lambda$:

$$\frac{\lambda}{2}\mathcal{L}^{AB}\mathcal{L}_{AB} = E^2 - \mathbf{P}^{\,2} - \frac{1}{R^2}\mathbf{L}^2 = m_{\Lambda 0}^2, \tag{25}$$

where $\mathcal{L}_{AB} = \eta_{AC}\eta_{BD}\mathcal{L}^{CD}$, $m_{\Lambda 0}$ introduced above should be the inertial-type mass for a free particle. It is well defined together with the energy, momentum and angular momentum at classical level.

It can further be shown that $m_{\Lambda 0}$ is the eigenvalue of the first Casimir operator of the dS group $\mathcal{G}_\Lambda$ [4]. In addition, spin can also be well defined as it was done in the relativistic quantum mechanics in Minkowski spacetime. We will explain the issue in detail elsewhere.

In any case, these offer a consistent way to define the observables for free particles and this kind of definitions differ from any others in dS space. Of course, these issues significantly indicate that the motion of a free particle in $\mathcal{B}_\Lambda$ should be of inertial-type in analog to Newton's and Einstein's conception for the inertial motion of a free particle with constant velocity. Consequently, the coordinate systems with Beltrami metric should be the globally inertial-type systems and corresponding observer at the origin of the system should be of inertial-type as well.

IV. ON DEFINITIONS OF SIMULTANEITY, SPACE-TIME MEASUREMENTS AND ROBERTSON-WALKER-LIKE METRIC

Among physical measurements, space-time measurements are most fundamental. In order to make space-time measurements, one should first define simultaneity. It is worth to note that there are two different kinds of definitions of simultaneity in the $\mathcal{B}_\Lambda$. Now, we discuss the two kinds of definitions separately.

In SR, coordinates have measurement significance that is linked with the SR principle. Namely, the difference in time coordinate stands for the time interval, and the difference in spatial coordinate stands for the spatial distance. Similar to Einstein's SR, one can define that two events A and B are simultaneous if and only if the Beltrami time coordinate x^0 for the two events are same,

$$a^0 := x^0(A) = x^0(B) =: b^0. \tag{26}$$

It is with respect to this simultaneity that free particles move along straight lines with uniform velocities. The simultaneity defines a 3+1 decomposition of spacetime

$$ds^2 = N^2(dx^0)^2 - h_{ab}\left(dx^a + N^a dx^0\right)\left(dx^b + N^b dx^0\right) \tag{27}$$

with the lapse function, shift vector, and induced 3-geometry on 3-hypersurface Σ_c in one coordinate patch.

$$\begin{aligned} N &= \{\sigma_{\Sigma_c}(x)[1-(x^0/R)^2]\}^{-1/2}, \\ N^a &= x^0 x^a [R^2 - (x^0)^2]^{-1}, \\ h_{ab} &= \delta_{ab}\sigma_{\Sigma_c}^{-1}(x) - [R\sigma_{\Sigma_c}(x)]^{-2}\delta_{ac}\delta_{bd}x^c x^d, \end{aligned} \tag{28}$$

respectively, where $\sigma_{\Sigma_c}(x) = 1 - (x^0/R)^2 + \delta_{ab}x^a x^b/R^2$, δ_{ab} is the Kronecker δ-symbol, $a, b = 1, 2, 3$. In particular, at $x^0 = 0$, $\sigma_{\Sigma_c}(x) = 1 + \delta_{ab}x^a x^b/R^2$. In a vicinity of the origin of Beltrami coordinate system in one patch, 3-hypersurface Σ_c acts as a Cauchy surface.

This simultaneity defines the laboratory time in one patch. According to the spirit of SR as well as the SR-type principle, the Beltrami coordinates define, in such a manner, the standard clocks and standard rulers in laboratory on $\mathcal{B}_\Lambda$. To measure the time of a process or the size of an object, we just need to compare with Beltrami coordinates.

There is another simultaneity, however. It is, in fact, with respect to the proper time of a clock rest at spatial origin of the Beltrami coordinate system. It can be shown that the proper time $\tau_{\Lambda>0}$ of a rest clock on the time axis of Beltrami coordinate system, $\{x^a = 0\}$, reads

$$\tau_{\Lambda>0} = R \sinh^{-1}(R^{-1}\sigma^{-\frac{1}{2}}(x)x^0). \tag{29}$$

Therefore, we can define that the events are simultaneous with respect to the proper time of a clock rest at the origin of the Beltrami spatial coordinates if and only if

$$x^0\sigma^{-1/2}(x,x) = \xi^0 := R\sinh(R^{-1}\tau) = \text{constant}. \tag{30}$$

The line-element on the simultaneous 3-d hypersurface, denoted by Σ_τ, can be defined as

$$dl^2 = -ds^2_{\Sigma_\tau}, \tag{31}$$

where

$$\begin{array}{l} ds^2_{\Sigma_\tau} = R^2_{\Sigma_\tau} dl^2_{\Sigma_\tau 0}, \\ R^2_{\Sigma_\tau} := \sigma^{-1}(x,x)\sigma_{\Sigma_\tau}(x,x) = 1 + (\xi^0/R)^2, \\ \sigma_{\Sigma_\tau}(x,x) := 1 + R^{-2}\delta_{ab}x^a x^b > 0, \\ dl^2_{\Sigma_\tau 0} := \{\delta_{ab}\sigma^{-1}_{\Sigma_\tau}(x) - [R\sigma_{\Sigma_\tau}(x)]^{-2}\delta_{ac}\delta_{bd}x^c x^d\}dx^a dx^b. \end{array} \tag{32}$$

Note that Σ_τ has a positive spatial curvature.

It should be pointed out that this simultaneity is closely linked with the cosmological principle. In fact, it is significant that if $\tau_{\Lambda>0}$ is taken as a "cosmic time", the Beltrami metric (7) becomes

$$ds^2 = d\tau^2 - dl^2 = d\tau^2 - R^2\cosh^2(R^{-1}\tau)dl^2_{\Sigma_\tau 0}. \tag{33}$$

It is the RW-like metric with a positive spatial curvature and the simultaneity is globally defined in whole $\mathcal{B}_\Lambda$.

It should be emphasized that the two definitions of simultaneity do make sense in different kinds of measurements. The first concerns the measurements in a laboratory and is related to the SR-type principle in $\mathcal{B}_\Lambda$ patch by patch, while the second concerns the observations with cosmological principle. Furthermore, the relation between the Beltrami metric and its RW-like correspondence (33) is meaningful. It links the coordinate time x^0 in laboratory and the cosmic time τ in a manifest way. This may shed light on the puzzle between laboratory time coordinate and cosmic time with arrow, another version of the SRP-CP puzzle. This also shows that the 3-d cosmic space is S^3 rather than flat. The deviation from the flatness is of order Λ. Obviously, this spatial closeness of the universe is another remarkable property different from the standard cosmological model with flatness. This property seems more or less already indicated by the CMB power spectrum from WMAP [11] and should be further checked by its data in large scale.

V. REMARKS

We have shown that in $\mathcal{B}_\Lambda$, the Beltrami coordinates should be regarded as inertial-type since the test particles and signals move along the timelike, null straight world lines, respectively. Therefore, their classical observables can be well defined. Consequently, the dS observable puzzle might be solved at least partly in such a way.

Furthermore, it is meaningful that the SRP-CP paradox might be solved by the relation between the Beltrami metric and the RW-like metric in $\mathcal{B}_\Lambda$. This relation links the coordinate time x^0 in the laboratory of one patch and the cosmic time τ in the large scale and shows that the 3-d cosmic space is slightly closed. This should be further checked by the data from WMAP [11] in large scale.

In fact all properties in $\mathcal{B}_\Lambda$ are in analog with SR and coincide with it if $\Lambda = 0$. Meanwhile, the SRP-CP paradox and other puzzles will appear again.

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